Math 4200
Wednesday November 4
3.3 Laurent series: classification of isolated singularities (Monday's notes) and multiplying Laurent series (today's notes).

Announcements: Quiz today!
Midterm next Friday will cover thru section 4.2 (The residue theorem).
Next homework assignment (due next Wednesday) is in today's notes.

Homework questions?

multiplying Laurent series term by term is legal:

We already know that we get the coefficient of $(z - z_0)^n$ in the Taylor series of a product $f(z)g(z)$ of analytic functions, by collecting the finite number of terms in the product of the Taylor series for f and g at z_0 which have that total power. The analogous statement is true for Laurent series, except that you may be collecting infinitely many terms. (You have a homework problem like this for Wednesday, 3.3.6.)

<u>Theorem</u> Let $f(z)$, $g(z)$ have Laurent series in $A := \{z \mid R_1 < |z - z_0| < R_2\}$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{m=1}^{\infty} b_m (z - z_0)^{-m} := \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

$$g(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k .$$

Then $f(z)g(z)$ has Laurent series

$$f(z)g(z) = \sum_{n=-\infty}^{\infty} d_n (z - z_0)^n$$

where

$$d_n = \lim_{N \to \infty} \sum_{j=-N}^{N} a_j\, c_{n-j}.$$

proof: Recall from Monday that we can recover the Laurent coefficients for an analytic function with a contour integral. Specifically, if γ is any p.w. C^1 contour in A, with $I(\gamma, z_0) = 1$, e.g. any circle of radius r, with $R_1 < r < R_2$, then Laurent coefficients for f are given by

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

$$b_m = \frac{1}{2\pi i} \int_\gamma f(\zeta) (\zeta - z_0)^{m-1} d\zeta,$$

which, if we write the Laurent series by combining the two sums as above,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

is the formula

$$a_n = \frac{1}{2\pi i} \int_\gamma f(\zeta) (\zeta - z_0)^{-n-1} d\zeta, \qquad n \in \mathbb{Z}$$

Thus, fixing n, the n^{th} Laurent coefficient d_n for $f(z)g(z)$ is given by

$$d_n = \frac{1}{2\pi i}\int_\gamma f(\zeta)g(\zeta)(\zeta - z_0)^{-n-1}d\zeta, \qquad n \in \mathbb{Z}.$$

Consider the truncated Laurent series for f, g,

$$f_N(z) := \sum_{j=-N}^{N} a_j (z - z_0)^j \quad , \quad g_N(z) = \sum_{k=-N+n}^{N+n} c_k (z - z_0)^k$$

which converge uniformly to f, g on the contour γ as $N \to \infty$, so

$$d_n = \lim_{N\to\infty} \frac{1}{2\pi i}\int_\gamma f_N(\zeta)g_N(\zeta)(\zeta - z_0)^{-n-1}d\zeta,$$

$$= \lim_{N\to\infty} \sum_{j,k=-N}^{N} \int_\gamma a_j(\zeta - z_0)^j c_k(\zeta - z_0)^k (\zeta - z_0)^{-n-1}d\zeta$$

by multiplication and term-by-term integration of the finite-sum truncated Laurent series. And picking off the non-zero integrals yields

$$d_n = \lim_{N\to\infty} \sum_{j=-N}^{N} a_j\, c_{n-j}.$$

Example (relates to hw problem 3.3.6):

a) The function $f(z) = \dfrac{e^{\frac{1}{z}}}{1 - z}$ is analytic for $0 < |z| < 1$. Find its residue at $z_0 = 0$.

b) Let γ be the circle of radiius $\dfrac{1}{2}$ centered at the origin. Find

$$\int_\gamma f(z)\, dz.$$

Computing residues In Chapter 4 we'll see lots of examples where we need to compute residues at isolated singularities z_0, because they are related to interesting contour integrals or to improper integrals on the real line. In fact section 4.1 is all about residue computation shortcuts in case the situation is complicated. (The residue computations in the section 3.3 homework for this week are mostly somewhat straightforward.)

In most cases the function of concern is a quotient and the reason for the singularity is that the denominator function has a zero at z_0. There's a scary-looking table on page 250 of our text - that will be provided on the next midterm - although I could ask you to verify certain entries in addition to using them. Here are two table entries, one easy, one a bit more complicated. We have everything we need to check these table entries, because we know how to multiply Taylor series and Laurent series:

1) Let $f(z) = \frac{g(z)}{h(z)}$ where $g(z_0) \neq 0$, $h(z_0) = 0$, $h'(z_0) \neq 0$. Prove that f has a pole of order 1, and

$$Res(f, z_0) = \frac{g(z_0)}{h'(z_0)}$$

2) Let $f(z) = \frac{g(z)}{h(z)}$ where $g(z_0) \neq 0$, $h(z_0) = h'(z_0) = 0$, $h''(z_0) \neq 0$. Then f has a pole of order 2 and

$$Res(f, z_0) = \frac{2\, g'(z_0)}{h''(z_0)} - \frac{2}{3} \frac{g(z_0) h'''(z_0)}{h''(z_0)^2} \qquad !!!$$

Math 4200-001
Homework 11
4.1-4.2
Due Wednesday November 11 at 11:59 p.m.
Exam will cover thru 4.2

4.1 1de, 3, 5, 7ab, 9
4.2 2 (Section 2.3 Cauchy's Theorem), 3, 4, 6, 9, 13.

w11.1 (extra credit) Prove Prop 4.1.7, the determinant computation for the residue at an order k pole for $f(z) = \frac{g(z)}{h(z)}$ at z_0, where $g(z_0) \neq 0$. (Hint: it's Cramer's rule for a system of equations.)